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EUGENIUSZ GŁOWACKI

On approximate solutions of linear functional equations

INTRODUCTION. Solving functional equations one usually obtains the solutions as the limits of some functional sequences formed with the aid of the given functions occuring in the equation. The terms of those sequences may be sums or products or combinations of sums and products built of given functions. In general it is difficult, and in many cases even impossible, to give the solution in a finite, explicit form. Therefore it appears as an important problem to find approximate solutions, i.e., to find functions differing from the exact solutions less than a given quantity. Of course, such a problem is meaningful only if there exists, for the equation considered, a unique solution or, among all the solutions, a unique solution fulfilling certain additional conditions. In order to guarantee the existence of such a unique solution, suitable conditions must be imposed on the given functions.

In the present paper we construct approximate solutions, with a preassigned accuracy, for continuous solutions of some linear functional equations in cases, where the equations have a unique continuous solution or a unique one-parameter family of continuous solutions.

We shall deal with the homogeneous linear equation

$$\varphi[f(x)] = g(x)\varphi(x), \tag{1}$$

and with the inhomogeneous linear equation

$$\varphi[f(x)] = g(x)\varphi(x)+F(x), \tag{2}$$

where $\varphi(x)$ is the required function. The values of the functions $\varphi(x)$, $g(x)$, $F(x)$ lie in the field $\mathcal{K}$ of the real or of the complex numbers, whereas $f(x)$ is a real-valued function of a real variable.

In the sequel we shall always assume that the function $f(x)$ fulfils the following condition:

(I) $f(x)$ is continuous and stricly increasing in an interval $[a, b)$, moreover $a<f(x)<x$ for $x \in (a, b)$.

Let $f^n(x)$ denote the n-th iterate of the function $f(x)$:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \quad n = 0, 1, 2, \ldots$$

Under hypothesis (I) we have $f(a)=a$ and the iterates $f^n(x)$ are defined, continuous and stricly increasing in $[a, b)$; furthermore, for every fixed $x \in (a, b)$ the sequence $\{f^n(x)\}$ is stricly decreasing and $\lim\limits_{n\to\infty} f^n(x)=a$ (cf. [1]).

Moreover, we shall make some of the following assumptions:

(II) The function $g(x)$ is continuous and $g(x)\neq 0$ for $x \in [a, b)$.

(III) The function $F(x)$ is continuous in $[a, b)$.

(IV) There exist constants $0<\Theta<1$, $0<\delta_1<b-a$ such that

$$f(x)-a\leqslant\Theta(x-a) \quad \text{for} \quad x\ [a, a+\delta_1]. \tag{3}$$

(V) There exist constants $A>0$, $0<\delta_2<b-a$, $\mu>0$ such that

$$|g(x)-1|\leqslant A(x-a)^{\mu} \quad \text{for} \quad x\in[a, a+\delta_2]. \tag{4}$$

(VI) There exist constants $\varrho>1$, $0<\delta_3<b-a$ such that

$$|g(x)|\geqslant\varrho \quad \text{for} \quad x\in[a, a+\delta_3]. \tag{5}$$

(VII) There exist constants $B>0$, $\nu>0$, $0<\delta_4<b-a$ such that

$$|F(x)|\leqslant B(x-a)^{\nu} \quad \text{for} \quad x\in[a, a+\delta_4]. \tag{6}$$

(VIII) There exist constants $C>0$, $\varkappa>0$, $0<\delta_5<b-a$ such that

$$|F(x)-F(a)|\leqslant C(x-a)^{\varkappa} \quad \text{for} \quad x\in[a, a+\delta_5]. \tag{7}$$

§ 1. We start with equation (1).

As has been proved in [2], under hypotheses (I), (II), (IV), (V) equation (1) has, for every $\eta \in \mathscr{K}$, a unique solution which is continuous in $[a, b)$ and fulfils the condition $\varphi(a)=\eta$. This solution is given by formula

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x), \tag{8}$$

where

$$\varphi_n(x) = \frac{\eta}{G_n(x)} \tag{9}$$

and

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]. \tag{10}$$

The function $\varphi_n(x)$ defined by (9) and (10) may be regarded as an approximate solution of equation (1).

Let us fix arbitrarily $\eta \in \mathscr{K}$, $0<d<b-a$, $\varepsilon>0$. We are going to find the index n in such a manner that the function $\varphi_n(x)$ should differ from the exact solution $\varphi(x)$ in the whole interval $[a, a+d]$ by less than ε.

If $\eta=0$, then $\varphi(x)\equiv 0$, $\varphi_n(x)\equiv 0$ and $|\varphi_n(x)-\varphi(x)|<\varepsilon$ for every n. So let $\eta\neq 0$ and let δ be an arbitrary positive number fulfilling the inequality

$$\delta<\min\left(\delta_1, \delta_2, \left(\frac{1-\Theta^{\mu}}{A}\right)^{1/\mu}\right). \tag{11}$$

It follows from (11) that for $x \in [a, a+\delta]$ inequalities (3) and (4) hold and moreover

$$(12) \qquad 0<\frac{\Theta^{\mu}}{1-A\delta^{\mu}}<1 .$$

Let N_1 be the least non-negative integer such that

$$(13) \qquad f^{N_1}(a+d)\in[a, a+\delta] .$$

Hence it follows by hypothesis (I) that

$$(14) \qquad f^{n}(x)\in[a, a+\delta] \quad \text{for} \quad n\geqslant N_1 \quad \text{and} \quad x\in[a, a+d] .$$

Let us write

$$(15) \qquad \Theta_1=\frac{\Theta^{\mu}}{1-A\delta^{\mu}},$$

$$(16) \qquad K=\inf_{[a,a+d]} g(x),$$

$$(17) \qquad L=\frac{|\eta| A\delta^{\mu}}{K^{N_1}(1-A\delta^{\mu})},$$

$$(18) \qquad \alpha=\max\left(N_1, N_1+\log_{\Theta_1}\frac{(1-\Theta_1)\varepsilon}{L}\right).$$

We have the following

THEOREM 1. *Under hypoteses (I), (II), (IV) and (V) for every $n>\alpha$ and every $x \in [a, a+d]$, we have*

$$(19) \qquad |\varphi_n(x)-\varphi(x)|<\varepsilon ,$$

where $\varphi_n(x)$ and $\varphi(x)$ are defined by (8), (9), (10).

Proof. In virtue of (14) we have for $n\geqslant N_1$ and $x \in [a, a+d]$

$$(20) \qquad |g[f^{n}(x)]-1|\leqslant A(f^{n}(x)-a),$$

$$(21) \qquad f^{n}(x)-a\leqslant\Theta^{n-N_1}[f^{N_1}(x)-a]\leqslant\Theta^{n-N_1}\delta,$$

$$(22) \qquad g[f^{n}(x)]\geqslant 1-A\delta^{\mu}>0.$$

By (9), (10), (16), (20), (21), (22) we get the estimation of $\varphi_{n+1}(x)-\varphi_n(x)$, valid for $x \in [a, a+d]$ and $n\geqslant N_1$

$$|\varphi_{n+1}(x)-\varphi_n(x)|=\frac{|\eta|\,|g[f^{n}(x)]-1|}{\prod_{i=0}^{N_1-1} g[f^{i}(x)]\prod_{i=N_1}^{n} g[f^{i}(x)]}\leqslant\frac{|\eta| A\delta^{\mu}}{K^{N_1}(1-A\delta^{\mu})}\,\frac{(\Theta^{\mu})^{n-N_1}}{(1-A\delta^{\mu})^{n-N_1}}.$$

In view of (15) and (17) we may write

$$|\varphi_{n+1}(x)-\varphi_n(x)|\leqslant L\Theta_1^{n-N_1} \quad \text{for} \quad n\geqslant N_1 \quad \text{and} \quad x\in[a, a+d].$$

By inequality (12) we have the following estimation of the difference $\varphi_n(x)-\varphi(x)$, valid for $x \in [a, a+d]$ and $n \geqslant N_1$

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{k=n}^{\infty} |\varphi_{k+1}(x)-\varphi_k(x)| \leqslant L \sum_{k=n}^{\infty} \Theta_1^{k-N_1} = \frac{L}{1-\Theta_1} \cdot \Theta_1^{n-N_1}.$$

Hence we get for $n>\alpha$, where α is defined by (18), and for all $x \in [a, a+d]$,

$$|\varphi_n(x)-\varphi(x)| < \varepsilon$$

which was to be proved.

It results from the estimations obtained in the proof that the convergence of the sequence $\{\varphi_n(x)\}$ is the more rapid the smaller is the number Θ_1, or, according to (15), the smaller is δ. But if δ decreases, then N_1 increases and consequently α need not decrease. We shall see this on the following example.

Example 1. For the equation

$$\varphi\left(\frac{x}{2+x}\right) = \frac{2+3x}{(2+x)(1+2x)} \varphi(x) \quad \text{in} \quad [0, \infty)$$

we have $a=0$, $b=\infty$, $f(x)=\dfrac{x}{2+x}$, $g(x)=\dfrac{2+3x}{(2+x)(1+2x)}$, $f^n(x)=\dfrac{x}{2^n+(2^n-1)x}$. We take $\eta=1$, $d=1$, $\varepsilon=10^{-3}$. We have $f^n(x) \leqslant \frac{1}{2}x$, $|g(x)-1| \leqslant x$ for $x \in [0, 1]$, $K = \inf_{[0,1]} g(x) = \frac{5}{9}$. Thus we may take $\Theta=\frac{1}{2}$, $\delta_1=\delta_2=1$, $A=1$, $\mu=1$. Inequality (10) turns then into $\delta<\frac{1}{2}$.

An easy calculation yields:

for $\delta=\dfrac{9}{20}$, $N_1=1$, $\Theta_1=\dfrac{10}{11}$, $L=\dfrac{81}{55}$, $102<\alpha<103$;

for $\delta=\dfrac{1}{3}$, $N_1=1$, $\Theta_1=\dfrac{3}{4}$, $L=\dfrac{9}{10}$, $30<\alpha<31$;

for $\delta=\dfrac{1}{60}$, $N_1=5$, $\Theta_1=\dfrac{30}{59}$, $L=\dfrac{9^4}{5^4 \cdot 59}$, $13<\alpha<14$;

for $\delta=\dfrac{1}{100}$, $N_1=6$, $\Theta_1=\dfrac{50}{99}$, $L=\dfrac{9^5}{5^6 \cdot 11}$, $15<\alpha<16$.

As we see, of the four cases considered $\delta=\frac{1}{60}$ yields the best result in the sense that the corresponding α is the smallest. This result cannot be much improved, since we have for our equation

$$\sup_{[0,1]} |\varphi_n(x)-\varphi(x)| \geqslant 10^{-3} \quad \text{for} \quad n<12.$$

Thus we can modify α by changing δ. Anyhow, in the general case we are unable to indicate the optimal choice of δ (which leads to the minimal α), since the dependence of α on δ is too complicated.

§ 2. Now we turn to equation (2). If hypoteses (I), (II), (III), (IV), (V), (VII) are fulfilled, then equation (2) has, for every $\eta \in \mathcal{X}$, a unique solution $\Phi(x)$ which is continuous in $[a, b)$ and fulfils the condition $\Phi(a)=\eta$ (cf. [2]). This solution is given as

(23) $$\Phi(x) = \lim_{n\to\infty} \Phi_n(x),$$

where

(24) $$\Phi_n(x) = \varphi_n(x) + \Psi_n(x),$$

$\varphi_n(x)$ is defined by (9) and (10), and

(25) $$\Psi_n(x) = -\sum_{i=0}^{n} \frac{F[f^i(x)]}{G_{i+1}(x)}.$$

We write

(26) $$\Psi(x) = \lim_{n\to\infty} \Psi_n(x).$$

Similarly as in the case of equation (1), we take as the approximate solution of equation (2) the function $\Phi_n(x)$.

Let us fix arbitrarily $\eta \in \mathcal{X}$, $0<d<b-a$, $\varepsilon>0$ and let us put $\varepsilon=\varepsilon_1+\varepsilon_2$, $\varepsilon_1>0$, $\varepsilon_2>0$. By theorem 1, to ε_1 a number α_1 may be chosen such that

(27) $$|\varphi_n(x)-\varphi(x)|<\varepsilon_1 \quad \text{for} \quad x\in[a, a+d], \quad n>\alpha_1.$$

Let $\bar{\delta}$ be an arbitrary positive number fulfilling the inequality

(28) $$\bar{\delta}<\min\left(\delta_1, \delta_2, \delta_4, \left(\frac{1-\Theta^{\nu}}{A}\right)^{1/\mu}\right).$$

It follows from the definition of $\bar{\delta}$ that for $x \in [a, a+\bar{\delta}]$ inequalities (3), (4), (6) hold and, moreover,

(29) $$0<\frac{\Theta^{\nu}}{1-A\bar{\delta}^{\mu}}<1.$$

Let N_2 be the least non-negative integer such that

(30) $$f^{N_2}(a+d)\in[a, a+\bar{\delta}].$$

Consequently, in virtue of hypotesis (I)

(31) $$f^n(x)\in[a, a+\bar{\delta}] \quad \text{for} \quad x\in[a, a+d] \quad \text{and} \quad n\geqslant N_2.$$

Let K be given by (16) and let us put

(32) $$\Theta_2 = \frac{\Theta^{\nu}}{1-A\bar{\delta}^{\mu}},$$

(33) $$L_1 = \frac{B\bar{\delta}^{\nu}}{K^{N_2}(1-A\bar{\delta}^{\mu})},$$

(34) $$\beta = \max\left(N_2, N_2+\log_{\Theta_2}\frac{(1-\Theta_2)\varepsilon_2}{L_1}\right).$$

We have the following

THEOREM 2. *Under hypoteses (I), (II), (III), (IV), (V) and (VII), for every* $n > \max(\alpha_1, \beta)$ *and* $x \in [a, a+d]$, *we have*

$$|\Phi_n(x) - \Phi(x)| < \varepsilon, \tag{35}$$

where $\Phi_n(x)$ *and* $\Phi(x)$ *are defined by* (23), (24), (25), (9), (10).

Proof. By (28), (3), (4), (6), and (31) we have for $n \geqslant N_2$ and $x \in [a, a+d]$

$$|F[f^n(x)]| \leqslant B[f^n(x) - a]^\nu, \tag{36}$$

$$f^n(x) - a \leqslant \Theta^{n-N_2}[f^{N_2}(x) - a] \leqslant \Theta^{n-N_2}\bar{\delta}, \tag{37}$$

$$g[f^n(x)] \geqslant 1 - A\bar{\delta}^\mu > 0. \tag{38}$$

By (25), (26), (16), (36), (37) and (38) we get the estimation for the difference $\Psi_n(x) - \Psi(x)$, valid for $x \in [a, a+d]$ and $n \geqslant N_2$,

$$|\Psi_n(x) - \Psi(x)| \leqslant \sum_{k=n+1}^{\infty} \frac{|F[f^k(x)]|}{\prod_{i=0}^{N_2-1} g[f^i(x)] \prod_{i=N_2}^{k} g[f^i(x)]} \leqslant \sum_{k=n+1}^{\infty} \frac{B\bar{\delta}^\nu(\Theta^\nu)^{k-N_2}}{K^{N_2}(1 - A\bar{\delta}^\mu)^{k-N_2+1}}.$$

With the notation (32), (33) we have according to inequality (29),

$$|\Psi_n(x) - \Psi(x)| \leqslant \sum_{k=n+1}^{\infty} L_1 \Theta_2^{k-N_2} = \frac{L_1}{1-\Theta_2} \Theta_2^{n-N_2+1} \quad \text{for} \quad x \in [a, a+d], \quad n \geqslant N_2.$$

For $n > \beta$, where β is defined by (34), we get hence the estimation

$$|\Psi_n(x) - \Psi(x)| < \varepsilon_2 \quad \text{for} \quad x \in [a, a+d]. \tag{39}$$

Now, for $n > \max(\alpha_1, \beta)$ inequalities (27) and (39) are both fulfilled and we get

$$|\Phi_n(x) - \Phi(x)| \leqslant |\varphi_n(x) - \varphi(x)| + |\Psi_n(x) - \Psi(x)| < \varepsilon$$

for $x \in [a, a+d]$, which was to be proved.

The manner how we split ε into the sum $\varepsilon_1 + \varepsilon_2$ depends on the rapidity of the convergence of sequences $\{\varphi_n(x)\}$ and $\{\Psi_n(x)\}$. This may be seen from the following example.

Example 2. We consider the functional equation

$$\varphi\left(\frac{x}{2}\right) = \left(1 + \frac{\sqrt[4]{x}}{10}\right)\varphi(x) + x^4, \quad x \in [0, \infty).$$

Here we have

$$f(x) = \frac{x}{2}, \; g(x) = \left(1 + \frac{\sqrt[4]{x}}{10}\right), \; F(x) = x^4, \; f^n(x) = \frac{x}{2^n}, \quad K = \inf_{[0,1]} g(x) = 1.$$

We take $\eta = 1$, $d = 1$, $\varepsilon = 10^{-2}$. For $x \in [0, 1]$ we have $f(x) \leqslant \frac{1}{2}x$, $|g(x) - 1| \leqslant \frac{1}{10}x^{1/4}$,

$F(x) \leqslant x^4$, and thus we may take $\Theta = \frac{1}{2}$, $A = \frac{1}{10}$, $B = 1$, $\mu = \frac{1}{4}$, $\nu = 4$, $\delta_1 = \delta_2 = \delta_4 = 1$. Inequalities (11) and (28) yield

$$\delta < 1, \quad \bar{\delta} < 1.$$

We take $\delta = \bar{\delta} = 2^{-12}$. Then $N_1 = N_2 = 12$ and

$$\Theta_1 = \frac{40\sqrt[4]{8}}{79}, \quad L = \frac{1}{79}, \quad \Theta_2 = \frac{5}{79}, \quad L_1 = \frac{5}{2^{44} \cdot 79}.$$

Consequently, the sequence $\{\Psi_n(x)\}$ converges considerably faster than does $\{\varphi_n(x)\}$. Let us consider two decompositions of ε into the sum $\varepsilon_1 + \varepsilon_2$.

I. $\varepsilon_1 = \frac{1}{200}$, $\varepsilon_2 = \frac{1}{200}$. Then we find after simple calculations

$$29 < \max(\alpha_1, \beta) < 30 .$$

II. $\varepsilon_1 = \frac{999}{1000000}$, $\varepsilon_2 = \frac{1}{1000000}$. We find

$$25 < \max(\alpha_1, \beta) < 26 .$$

As we see, in our case the choice $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$ gives a worse effect than the choice of unequal $\varepsilon_1, \varepsilon_2$ which takes into account the faster convergence of the sequence $\{\Psi_n(x)\}$.

§ 3. In both cases considered so far we had $g(a) = 1$. Now we are going to deal with equation (2) in the case where $|g(a)| > 1$.

Let hypotheses (I), (II), (III), (VI) be fulfilled. Then equation (2) has a unique continuous solution $\varphi(x)$ in $[a, b)$ (cf. [1]). This solution is given by the formula

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x), \tag{40}$$

where

$$\varphi_n(x) = -\sum_{i=0}^{n} \frac{F[f^i(x)]}{G_{i+1}(x)} \tag{41}$$

and $G_n(x)$ is defined by (10).

Let us fix arbitrarily $\varepsilon > 0$ and $0 < d < b - a$. Let N_3 be the least non-negative integer such that

$$f^{N_3}(a+d) \in [a, a+\delta_3]. \tag{42}$$

It follows by hypothesis (I) that

$$f^n(x) \in [a, a+\delta_3] \quad \text{for} \quad x \in [a, a+d] \quad \text{and} \quad n \geqslant N_3. \tag{43}$$

We write

$$M = \sup_{[a,a+d]} |F(x)|, \qquad R = \inf_{[a,a+d]} |g(x)|, \tag{44}$$

$$\gamma = \max\left(N_3,\ N_3 - 1 + \log_\varrho \frac{M}{R^{N_3}(\varrho - 1)\varepsilon}\right). \tag{45}$$

Then we have the following

THEOREM 3. *Under hypotheses (I), (II), (III) and (VI), for every $n>\gamma$ and $x \in [a, a+d]$, we have*

$$|\varphi_n(x)-\varphi(x)|<\varepsilon,$$

where $\varphi_n(x)$, $\varphi(x)$ are defined by (40), (41) *and* (10).

Proof. By (43) and (5) we have for $n \geqslant N_3$ and $x \in [a, a+d]$

$$|g[f^n(x)]| \geqslant \varrho. \tag{46}$$

Making use of (40), (41), (44) and (46) we get the estimation, valid for $n \geqslant N_3$ and $x \in [a, a+d]$

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{k=n+1}^{\infty} \frac{|F[f^k(x)]|}{\prod_{i=0}^{N_3-1} |g[f^i(x)]| \prod_{i=N_3}^{k} |g[f^i(x)]|} \leqslant \sum_{k=n+1}^{\infty} \frac{M}{R^{N_3}\varrho^{k-N_3+1}}.$$

Since $0<1/\varrho<1$, we obtain further

$$|\varphi_n(x)-\varphi(x)| \leqslant \frac{M}{R^{N_3}(\varrho-1)\varrho^{n-N_3+1}} \quad \text{for} \quad x \in [a, a+d], \quad n \geqslant N_3.$$

Hence it follows, in virtue of the definition (45) of the number γ,

$$|\varphi_n(x)-\varphi(x)|<\varepsilon \quad \text{for} \quad x \in [a, a+d] \quad \text{and} \quad n>\gamma,$$

which was to be proved.

Example 3. We consider the equation

$$\varphi\left(\frac{x}{\sqrt{2+x^2}}\right) = \frac{1-x^2}{1+x^2}\varphi(x)+x^2 \quad \text{in} \quad [0, 2).$$

Here we have $f(x) = \dfrac{x}{\sqrt{2+x^2}}$, $g(x) = \dfrac{4-x^2}{1+x^2}$, $F(x) = x^2$. Let $d=\frac{3}{2}$, $\varepsilon=10^{-3}$. Since for $x \in [0, \frac{1}{2}]$ we have $|g(x)| \geqslant 3$, we may take $\varrho=3$, $\delta_3=\frac{1}{2}$. We have

$$R = \inf_{[0,3/2]} |g(x)| = \frac{7}{13}, \qquad M = \sup_{[0,3/2]} F(x) = \frac{9}{4}, \qquad N_3 = 2,$$

and we get the estimation for γ:

$$7<\gamma<8.$$

Thus as the approximate solution we may take e.g.

$$\varphi_8(x) = -\sum_{k=0}^{8} \frac{x^2}{(1+x^2)^{k+1}[2^k+(2^k-1)x^2]} \prod_{i=0}^{k} \frac{2^i}{[2^{i+2}+(2^{i+2}-5)x^2]}$$

and we have the accuracy

$$|\varphi_8(x)-\varphi(x)|<10^{-3} \quad \text{for} \quad x\in[0,\tfrac{3}{2}].$$

In the case of theorem 3, γ decreases with the increase of ϱ, the remaining parameters being fixed. The number ϱ may be made bigger, e.g. when the function $|g(x)|$ is decreasing in a neighbourhood of a, by taking a smaller δ_3. But then the number N_3 increases and consequently γ need not decrease.

§ 3. Now we shall deal with two particular cases of equation (2), viz., those where $g(x)\equiv 1$ or $g(x)\equiv -1$.

Thus we consider the equations

$$\varphi[f(x)]+\varphi(x)=F(x) \tag{47}$$

and

$$\varphi[f(x)]-\varphi(x)=F(x). \tag{48}$$

As has been proved in [1], if hypotheses (I), (III), (IV), and (VIII) are fulfilled, then equation (47) has a unique continuous solution in $[a,b)$; and if, moreover, $F(a)=0$, then equation (48) has, for every $\eta\in\mathscr{K}$, a unique solution $\varphi(x)$ which is continuous in $[a,b)$ and fulfils the condition $\varphi(a)=\eta$. In both cases these solutions may be written as

$$\varphi(x)=\lim_{n\to\infty}\varphi_n(x) \tag{49}$$

where

$$\varphi_n(x)=\sigma-\sum_{k=0}^{n}\omega^{k+1}\{F[f^k(x)]-F(a)\} \tag{50}$$

and

$$\sigma=\tfrac{1}{2}F(a),\quad \omega=-1 \qquad \text{for equation (47)},$$

$$\sigma=\eta,\quad \omega=1,\quad F(a)=0 \qquad \text{for equation (48)}.$$

Let us fix arbitrarily $\varepsilon>0$ and $0<d<b-a$. We put

$$\tilde{\delta}=\min(\delta_1,\delta_5). \tag{51}$$

Let N be the least non-negative integer such that

$$f^N(a+d)\in[a,a+\tilde{\delta}]. \tag{52}$$

Then, in virtue of hypothesis (I),

$$f^n(x)\in[a,a+\tilde{\delta}] \quad \text{for} \quad x\in[a,a+d] \quad \text{and} \quad n\geqslant N. \tag{53}$$

Further, we write

$$\lambda=\max\left(N,\,N-1+\log_{\Theta^\times}\frac{\varepsilon(1-\Theta^\times)}{C\tilde{\delta}^\times}\right). \tag{54}$$

Then we have the following

THEOREM 4. *Under hypotheses (I), (III), (IV) and (VIII) for every $n>\lambda$ and $x \in [a, a+d]$, we have*

$$|\varphi_n(x)-\varphi(x)|<\varepsilon,$$

where $\varphi_n(x)$, $\varphi(x)$ are given by (50) *and* (49).

Proof. By (53), (7), (51) and (3) we have for $n \geqslant N$ and $x \in [a, a+d]$

$$|F[f^n(x)]-F(a)| \leqslant C[f^n(x)-a]^{\varkappa}, \tag{55}$$

$$f^n(x)-a \leqslant \Theta^{n-N}[f^N(x)-a] \leqslant \Theta^{n-N}\tilde{\delta}. \tag{56}$$

Making use of inequalities (55), (56) and relations (49), (50), we obtain the estimation, valid for $x \in [a, a+d]$ and $n \geqslant N$,

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{k=n+1}^{\infty} |F[f^k(x)]-F(a)| \leqslant \sum_{k=n+1}^{\infty} C\tilde{\delta}^{\varkappa}(\Theta^{\varkappa})^{k-N}.$$

The last series is a convergent geometric series, whence

$$|\varphi_n(x)-\varphi(x)| \leqslant \frac{C\tilde{\delta}^{\varkappa}}{1-\Theta^{\varkappa}}(\Theta^{\varkappa})^{n+1-N} \quad \text{for} \quad x \in [a, a+d] \quad \text{and} \quad n \geqslant N.$$

We obtain hence for $x \in [a, a+d]$ and $n>\lambda$, where λ is defined by (54),

$$|\varphi_n(x)-\varphi(x)|<\varepsilon,$$

which was to be proved.

Example 4. We take the equation

$$\varphi\left(\frac{x}{2}\right)+\varphi(x)=\frac{1+x^2}{1-x^2} \quad \text{in} \quad [0,1).$$

Here we have $f(x)=\frac{x}{2}$, $F(x)=\frac{1+x^2}{1-x^2}$, $F(0)=1$; $f^n(x)=\frac{x}{2^n}$. We take $d=\frac{3}{4}$, $\varepsilon=10^{-4}$. For $x \in [0, \frac{1}{2}]$ we have the estimations

$$f(x) \leqslant \frac{1}{2}x, \quad |F(x)-1|=\frac{2x^2}{1-x^2} \leqslant \frac{8}{3}x^2.$$

Thus we may take $\tilde{\delta}=\frac{1}{2}$, $\Theta=\frac{1}{2}$, $C=\frac{8}{3}$, $\varkappa=2$. Then $N=1$ and we get

$$6<\lambda<7.$$

Consequently, as the approximate solution differing by less than 10^{-4} in the interval $[0, \frac{3}{4}]$ from the exact one, we may take e.g.

$$\varphi_7(x)=\frac{1}{2}+2x^2\sum_{k=0}^{7}\frac{(-1)^k}{2^{2k}-x^2}.$$

§ 5. In all cases considered for equations (1) and (2) the convergence of $\varphi_n(x)$ to $\varphi(x)$ is of geometric type:

$$|\varphi_n(x)-\varphi(x)| = O(s^n) \quad \text{for} \quad n\to\infty \quad \text{where} \quad 0<s<1.$$

In such cases the speed of the convergence increases when s decreases.

In theorems 1 and 2 we may improve the estimation of the rapidity of the convergence by a suitable choice of δ. In theorems 3 and 4 the number s is determined by the given functions appearing in the equation and in general the estimation cannot be improved.

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O PRZYBLIŻONYCH ROZWIĄZANIACH LINIOWYCH RÓWNAŃ FUNKCYJNYCH

Streszczenie

W pracy są rozważane przybliżenia ciągłych rozwiązań liniowych jednorodnych równań funkcyjnych postaci $\varphi[f(x)] = g(x)\varphi(x)$ oraz równań liniowych niejednorodnych postaci $\varphi[f(x)] = g(x)\varphi(x)+F(x)$, gdzie φ jest funkcją niewiadomą.

Dokładne rozwiązania ciągłe równań liniowych, w przypadkach jednoznaczności, są granicami pewnych ciągów funkcyjnych $\{\varphi_n(x)\}$ utworzonych za pomocą funkcji danych. Jako rozwiązanie przybliżone przyjmuje się n-ty wyraz tego ciągu.

Celem pracy jest dobór liczby rzeczywistej a w ten sposób, aby n-ty wyraz ciągu, $\varphi_n(x)$, różnił się od rozwiązania dokładnego (granicy tego ciągu) o z góry zadaną liczbę, w zadanym przedziale, gdy tylko $n>a$.

Praca zawiera 4 twierdzenia podające sposób doboru liczby a w różnych przypadkach równań liniowych. W każdym przypadku sposób doboru jest zilustrowany na odpowiednim przykładzie.

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